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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

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Hence, since a double-quadratic involves 9 parameters, it follows that *the double-quadratics having given group-quadratics*

$$P_x\ P'_y\ Q_x\ Q'_y\ R_x\ R'_y$$

form a linear three-parameter family. Obviously the double-quadratics $P_xP'_y$, $Q_xQ'_y$, $R_xR'_y$ are degenerate members of this family; for (PP′) is one pair of group-quadratics of $(P_xP'_y)$ and (QQ′); (RR′) may be supposed to be the other pairs, since for $P_xP'_y$ these other pairs are arbitrary except in so far as they are limited by conditions of apolarity. Hence, taking account of the number of parameters, we have the *Theorem*:

The general double-quadratic $F(x, y)$ which has the pairs of group-quadratics $P_xP'_y$, $Q_xQ'_y$, $R_xR'_y$ can be written in the form

$$F \equiv a\,P_xP'_y + b\,Q_xQ'_y + c\,R_xR'_y. \quad \dots \qquad \dots \quad (1)$$

This form will be called the *normal form* of F, when the group-quadratics PQR P′Q′R′ are all written with unit discriminants.* Under this supposition, the branch quartics of (1) are

$$\delta_x = a^2P_x^2 + b^2Q_x^2 + c^2R_x^2$$

$$\delta_y = a^2P'^2_y + b^2Q'^2_y + c^2R'^2_y. \quad \dots \qquad \dots \quad (2)$$

Remembering that PQR, P′Q′R′ are both apolar triads of quadratics with unit discriminants, it is obvious from these equations without any calculation that the absolute invariants of δ_x and δ_y should be equal, being the same functions of a, b, c.† We thus reach the fundamental *Theorem.*

The two branch quartics of a double-quadratic are linearly transformable into one another.

---

* The case when either of the branch-quartics has repeated roots has been excluded from the foregoing theory, so that none of the group-quadratics can be perfect squares.

† The absolute invariant of a quartic $f(x)$ may be regarded as a quantity defined by the position of $f(x)$ relative to the perfect squares in its standard pencil. For instance it is well-known that $f(x)$ and $\phi(\lambda)$ have the same absolute invariants, where the roots of $\phi(\lambda)$ are the parameters of $f(x)$ and the perfect squares, in the standard pencil of f. By using this theorem it would be easy to exhibit the absolute invariants of δ_x, δ_y as equal to the absolute invariant of the *same* quartic $\phi(\lambda)$.

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As immediate geometrical illustrations of this theorem, we may mention that

(1) The cross ratio of the common-points of two conics on one of them is equal to the cross ratio of the contact-points of their common tangents with the other.

(2) The cross ratio of the four tangents to a plane cubic from any point on it is constant.

(3) The cross-ratio of the four tangents to a binodal quartic from either of its nodes is the same.

(4) The cross ratio of the four generators of either system of a quadric, which touch another quadric, is the same.*

Another simple theorem which can be deduced from the normal form is the following :

An unsymmetric double-quadratic $F(xy)$ can be rendered symmetric by a linear transformation of one of the variables x or y alone.

For, since PQR, P′Q′R′ are apolar triads of quadratics with unit discriminants, there exists a transformation of y which transforms

$$P'_y,\ Q'_y,\ R'_y \text{ into } \pm KP_\eta,\ \pm KQ_\eta\ \pm KR_\eta.$$

This transformation renders F symmetric in x and η.

Let us examine now the family of double-quadratics with *given branch-quartics* δ_x, δ_y. Clearly if

$$F(xy) = a\, P_x P'_y + b\, Q_x Q'_y + c\, R_x R'_y$$

is to have always the same branch-quartics, $a\, b\, c$ must vary in such wise as to keep the forms

$$\delta_x = a^2 P_x^2 + b^2 Q_x^2 + c^2 R_x^2 ;$$
$$\delta_y = a^2 P'^2_y + b^2 Q'^2_y + c^2 R'^2_y$$

* This linear equivalence seems to have been first noticed by Cayley. Of the four geometrical illustrations, I find that the last has been treated by Cayley and the others have been mentioned and proved by Zeuthen ('Deductions de Theoremes Geometrique d'un seul Principe Algebrique.' *Proc. L.M.S.*, Vol. X, First Series). An extension of the theorem of equivalence which identifies the absolute invariant of $\delta_x\ \delta_y$ as that of a certain quaternary characteristic determinant is proved by Kasner (*loc cit*) and will be proved later on in this paper in a simple geometrical way.

constant multiples of themselves. Hence, taking account of the identities

$$P_x^2 + Q_x^2 + R_x^2 = P'^2_y + Q'^2_y + R'^2_y \equiv 0.$$

one sees that a, b, c must be connected by a relation of the form

$$\lambda a^2 + \mu b^2 + \nu c^2 = 0 \text{ where } \lambda + \mu + \nu = 0.$$

We conclude then, that the family Γ of forms with given branch quartics δ_x, δ_y is a *quadratic* family with one effective parameter and therefore, (what is more to our purpose) that it is a *unicursal* family, *i.e.* its members can be represented uniquely and reversibly by a parameter.

Turning our attention for a moment to the family of forms

$$a\, P_x P'_y + b\, Q_x Q'_y + c\, R_x R'_y$$

with given group-quadratics, let us enquire whether it can contain any perfect squares. Now a double-quadratic which is a perfect square must have both its branch quartics indeterminate. Hence $\Sigma a\, P_x P'_y$ will be a perfect square, if

$$a^2 P_x^2 + b^2 Q_x^2 + c^2 R_x^2 \equiv 0$$

and therefore a multiple of

$$P_x^2 + Q_x^2 + R_x^2,$$

i.e. if $$a^2 = b^2 = c^2.$$

Hence the family contains precisely *four* perfect squares

$$P_x P'_y \pm Q_x Q'_y \pm R_x R'_y.$$

It is obvious also, that on equating to zero any one of these perfect squares we would obtain a linear transformation which transforms

$$P_x,\ Q_x,\ R_x \text{ into } \pm KP'_y,\ \pm KQ'_y,\ \pm KR'_y$$

respectively where K is numerical.

Coming now to the family Γ of forms with given branch quartics δ_x, δ_y we see firstly *that* Γ *contains four perfect squares* (since $a^2 = b^2 = c^2$ satisfies $\lambda a^2 + \mu b^2 + \nu c^2 = 0$, in virtue of $\lambda + \mu + \nu = 0$) and secondly that these four perfect squares equated to zero give precisely the four linear transformations which transform δ_x into a multiple of δ_y. Representing the forms of Γ by a parameter t, we can show ***that if $f(t)$ represent parametrically the four perfect squares, then the***

quartics $f(t)$, δ_x, δ_y *have the same absolute invariants.** We can prove this by a special parametric representation. Let α be a root of δ_x and let t be the repeated root of $F(\alpha, y) = 0$ where F is any member of Γ. Then t may be taken as the parameter of F. Also since the four perfect squares represent linear transformations of δ_x into δ_y, it follows that their parameters would be roots of δ_y. Under this special parametric representation, then, $f(t)$ becomes a multiple of δ_y thus proving the theorem.

3. In the foregoing theory, the branch quartics δ_x and δ_y have been supposed to be quite general. The cases in which they are specialised are not of great importance in the general theory of the (2,2) correspondence and may be dismissed here with a few remarks.

(1) δ_x and δ_y are equi-anharmonic.

δ_x, δ_y and therefore also F will now admit a much larger group, *viz.* a tetrahedral group of order 12. The normal form and the other results proved would continue to be valid.

(2) δ_x and theretore also δ_y have a single repeated root.

Two of the group quadratics will now become identical and the normal form obtained would not be valid.

(3) δ_x and δ_y have a twice-repeated root.

The normal form is not valid. δ_x and δ_y admit now continuous one-parameter groups $[\gamma]$, $[\gamma']$ between the members of which there subsists a simple isomorphism in accordance with the equation

$$F[x, y] \equiv F[\gamma(x), \gamma'(y)]$$

(4) δ_x and δ_y are perfect squares p_x^2, p_y^2.

Same remarks as in (3) except that now the normal form is valid, F being in fact a multiple of $p_x p'_y$.

The (2,2) Correspondence.

4. The symbol F in addition to denoting the double-quadratic $F(xy)$ may also be used operationally to denote the (2,2) correspondence

* This theorem is, in substance though not in form, equivalent to Frobenius' extension of Cayley's theorem of the linear equivalence of δ_x, and δ_y proved by Kasner (*loc. cit.*)

or transformation defined by $F(xy) = 0$. The roots of $\delta_x = 0$, $\delta_y = 0$ will be called the x- and y-branch points of the transformation F. When $F(xy)$ is a perfect square, the transformation F will be termed a *homography*. The theorem

$$F(xy) \equiv F(I_r(x), I'_r(y))$$

may be stated thus:

If x, y are correspondents in F, so are $I_r(x)$, $I'_r(y)$

or in symbols

$$F = I'_r F I_r.$$

If y_1, y_2 are the correspondents of x in F, the correspondents of $I_1(x)$ would be $I'_1(y_1)$, $I'_1(y_2)$. Hence the eliminant of $F(xy)$ and any quadratic in x apolar to P_x is the product of two quadratics in y apolar to P'_y, and *vice versa*. Representing the quadratics apolar to P_x, P'_y respectively by parameters s, t, we see then, that $F(x, y) = 0$ implies a double-quadratic relation of the form $\phi(s, t) = 0$. ϕ will be termed *the derived transformation* of F corresponding to the group quadratics PP′. The branch s-elements of ϕ are the squares of the linear factors of P_s and the quadratic factors of δ_x which are apolar to P_x; the branch t-elements may be similarly specified.

It will be noticed that if

$$F_1(xy) \equiv F[I_1(x), y] \equiv F[x, I'_1(y)],$$

then the transformations F, F_1 have the same derived transformation corresponding to PP′.

It is also obvious that *one* pair of corresponding group quadratics of ϕ is the pair p_s, p'_t which represent parametrically the quadratics Q_xR_x, $Q_y'R_y'$ respectively. Call p'_s, p'_t the *special* group qudratics of ϕ.

Also, it has been mentioned that the eliminant of $F(xy)$ and any standard x-quartic is the product of two standard y-quartics. Representing standard x- and y-quartics by parameters λ, λ', we see then that $F(xy) = 0$ implies a double-quadratic relation of the form $\psi(\lambda, \lambda') = 0$ ψ will be called *the λ-transformation of F*. It is obvious that the four

transformations F, F_1, F_2, F_3 have all the same* λ-transformation, where

$$F_r(xy) \equiv F\,[I_r(x),\, y] \equiv F\,[x, I'_r(y)] \qquad (r = 1,2,3).$$

The branch-elements of ψ are the branch-quartics and the squares of the group-quadratics of F

If ϕ be the derived transformation of F corresponding to a pair of group-quadratics, (say PP′), then ψ the λ-transformation of F is also the derived transformation of ϕ corresponding to *its special group-quadratics.* For, any standard x-quartic α_x is the product of two factors apolar to P_x, which could be represented parametrically by a quadratic q_s apolar to p_s a special group-quadratic of ϕ. A similar consideration applied to the two standard y-quartics which are the transforms of α_x, puts the theorem in evidence.

It is important to observe that the branch-elements of $\phi_1 \phi_2 \phi_3$ the derived transformations of F and of ψ the λ-transformation of F, all depend only on the branch-points of F. Hence when F varies without changing its branch-points, so do ϕ_1, ϕ_2, ϕ_3 and ψ. We observe also in addition, that when F becomes a homography, so do ϕ_1, ϕ_2, ϕ_3, and ψ.

The Sub-rational Correspondence.

5. The rank of the matrix of the co-efficients of $F(x, y)$ is clearly a numerical invariant of F under simultaneous linear transformations of x, y.

If this rank is unity, F reduces to the product of two binary quadratics in x and y respectively.

If this rank is two, and $F(xy) = Lx_1^2 + 2M\,x_1x_2 + Nx_2^2$, there must be a identical linear relation between L, M and N; so that the equation $F(xy) = 0$ can be reduced to the form $T(x) = S(y)$ where T, S

* The sameness here refers to sameness apart from the particular parametrical representation chosen, *i.e.* to the sameness of linear transformability. The distinction does not arise in the text where the transformations are defined and handled in a concrete way.

are rational functions. The correspondence F will in this case be said to be *sub-rational.**

The normal form of a sub-rational correspondence F may be easily deduced from the equation $T(x) = S(y)$. Let I_1, I'_1 be the involutions determined by the numerator and denominator of T, S respectively. Then

$$T[I_1(x)] = T(x) = S(y) = S[I'_1(y)].$$

Hence $I_1 I_1'$ are a pair of group-involutions of F corresponding to the group quadratics P, P′ (say). Then the numerator and denominator of T, S would be linear combinations of the other group-quadratics, so that $T(x) = S(y)$ reduces to

$$\frac{Q_x}{R_x} = \frac{a\,Q'_y + b\,R'_y}{c\,Q'_y + d\,R'_y}$$

Using now the fact that (QQ′) and (RR′) are group-quadratics, we see immediately that $a\,Q'_y + b\,R'_y$ should be apolar to Q'_y and $c\,Q'_y + d\,R'_y$ to R'_y, giving $a = d = 0$. Hence if F is sub-rational, the normal form of F (xy) is

$$F(xy) = c\,Q_x Q'_y - b\,R_x R'_y.$$

The excluded group-quadratics (PP′) will be called the *relevant quadratics* of the sub-rational transformation F.

Theorem: If F be a sub-rational transformation, then

(1) ϕ_1, *the derived transformation of F corresponding to the relevant quadratics, is a homography.*

(2) *The* λ*-transformation of* F *is a homography.*

(3) ϕ_2 *and* ϕ_3 *the other derived transformations of F are both sub-rational, the relevant quadratics being in both cases the special group-quadratics.*

Let P, P′ be the relevant quadratics of F. The normal form of F shews that the correspondents $y_1 y_2$ in F, of any x are the roots of a quadratic apolar to P'_y. Hence x and $I_1(x)$ have exactly the same correspondents in F, which proves that the derived transformation ϕ_1

* The term sub-rational is borrowed from Sylvester who used it to denote the functional relation between two quantities x, y connected by the relation $\lambda(x) = \mu(y)$, λ, μ denoting rational functions.

corresponding to PP′ is a homography. The λ-transformation of F being a derived transformation of ϕ_1 must also be a homography (§ 4). It is obvious that ϕ_2, ϕ_3 the other derived transformations must be sub-rational. That the relevant quadratics of $\phi_2\phi_3$ are their special group-quadratics follows from the fact that their derived transformation with respect to the latter (which was proved to be the λ-transformation of F) is a homography.

Consider now a family Γ of transformations with fixed branch points. The co-efficients a, b, c of the normal form

$$a P_x P'_y + b Q_x Q'_y + c R_x R'_y$$

of the family were shown to be connected by a quadratic relation of the form

$$\lambda a^2 + \mu b^2 + \nu c^2 = 0 \text{ where } \lambda + \mu + \nu = 0.$$

Hence the family Γ contains *six* subrational transformations whose equations are of the type

$$\sqrt{\nu}\, Q_x Q'_y \pm \sqrt{-\mu}\, R_x R'_y = 0$$

If F be one of these six transformations, with the relevant quadratics (PP′), then the correspondents of any x in F are the roots of a quadratic apolar to P'_y and therefore the correspondent of a x-branch-point in F is a root of P'_y. Hence if we represent the transformations of Γ parametrically by the correspondent of a fixed x-branch-point in each transformation, it follows that the four homographies of Γ will be represented by S_y and the six sub-rational transformations by the product $P'_y Q'_y R'_y$. We thus reach the theorem :

The sextic representing parametrically the six sub-rational transformations of a family Γ with fixed branch-points, is the sextic covariant of the quartic representing the four homographies of Γ.

Ex. (1). If there exist two distinct values $\eta\eta'$ of y such that the equations $F(x\eta) = 0$, $F(x\eta') = 0$ have the same roots in x, prove that F is sub-rational.

Ex. (2). If F = LM where L, M are bilinear forms, shew that F is sub-rational only if L, M are apolar { *i.e.* if $(\alpha\alpha')(\beta\beta') = 0$ where $L = \alpha_x \beta_y$, $M = \alpha'_x \beta'_y$ }.

The (2,2) Curve on the Quadric.

6. The elliptic quartic curve F in space admits a pencil $\lambda S + S'$ of quadrics passing through it. The common self-polar tetrahedron of quadrics of this pencil may be called the *self-polar tetrahedron of F*. If the values of λ for which $\lambda S + S'$ is a cone are the roots of $f(\lambda)=0$ then the absolute invariant of $f(\lambda)$ is the *absolute invariant of the curve F*.

An elliptic quartic curve F on a quadric S, meets the generators of S twice and thereby determines a (2,2) correspondence $F(xy) = 0$, x, y representing parametrically generators of different systems meeting on F. The branch-elements of the correspondence are the two sets of four generators of each system which touch the curve. The cross ratio of these sets is therefore the same (Cayley's Theorem). Again it is geometrically obvious that the x- and y-involutions which transform these branch-generators into themselves are derivable from the three involutoric space-collineations determined by pairs of opposite edges of the self-polar tetrahedron of F. Therefore the curves on S which give rise to a family of transformations with given group-quadratics are the curves which have a common self-polar tetrahedron; and the four homographies of this family correspond to the cases in which the quartic curve degenerates into a repeated conic, namely, the conics of intersection of S with the faces of the self-polar tetrahedron.

Again, the curves which correspond to transformations with the same branch elements as F are the curves F′ which touch the same eight generators as F. These curves F′ are also the curves of contact of S with the common tangent developables of S and quadrics S′ passing through F. There is thus a (1,1) correspondence between quadrics S′ through F and transformations F′ having the same branch-generators as F. Hence

the absolute invariant of the curve F

= the absolute invariant of the four cones of the family (S′)

= the absolute invariant of the four homographies of the family (F′)

= the absolute invariant of either set of branch-generators of F (by the Theorem of § 2.)

This is in effect Frobenius' extension of Cayley's Theorem (Kasner : *loc. cit*).

Coming now to the geometrical meaning of sub-rational transformations, we see that the transformation F′ is sub-rational when the curve F′ admits inscribed quadrilaterals whose sides are generators of S. Clearly the planes determined by adjacent sides of such quadrilaterals would be tangent to S′ the corresponding quadric of F′ through F. Let us for convenience say that a quadric S′ is sub-rational to S when the curve of contact of S with their common tangent developable determines a sub-rational transformation on S. S′ then, would be sub-rational to S when there exist circumscribed tetrahedra of S′ having *two* pairs of opposite edges generators of S. Using now the theorem of § 5, we see immediately that *the quadrics S′ through F, sub-rational to S are given parametrically by the sextic covariant of the quartic representing the cones through F.* It is important to observe that this determination of S′ depends only on the parameters of the cones and not on the parameter of S. Hence if S′ is sub-rational to S, then S′ is sub-rational to *every* quadric λ S′ + S.*

As an exercise let us determine the quaternary invariant condition for the sub-rationality of S′ to S. This condition is that 0, the parameter of S′ in the pencil λ S + S′ should be a root of the sextic covariant of the quartic giving the cones, *viz.*

$$\Delta\lambda^4 + \theta\lambda^3 + \phi\lambda^2 + \theta'\lambda + \Delta' = 0.$$

This equation then should be capable of being transformed into itself by a transformation of the form $x = \dfrac{y}{py - 1}$. On actually making the transformation, we find two independent conditions.

$$2\theta + 2p\phi + p^3\theta' = 0$$
$$\theta' + 2p\Delta' = 0.$$

The elimination of p gives the usual invariant equation

$$8\theta\Delta'^2 + \theta'^3 = 4\Delta'\theta'\phi. \dagger$$

This equation has the disadvantage of concealing the fact that the S′ determined by it has an invariant relation to the pencil, and not merely an invariant relation to S.

* So far as I am aware, this does not seem to have been noticed before.

† Salmon remarks that this is analogous to the case of inscribed circumscribed triangles to two conics *in plano*. As a matter of fact, the analogy is with the case of inscribed-circumscribed quadirlaterals, the transformations, in both cases being sub-rational.

Ex. (3) Prove by reciprocation or otherwise, that if the curve of intersection of two quadrics determines a subrational correspondence on one of them, the curve of contact of the common tangent developable determines a sub-rational correspondence on the other.

The Symmetric Correspondence.

7. If the double-quadratic $F(xy)$ is symmetric in x, y, it would be thereby implied that the variables x, y are co-gredient. There would then exist an identical concomitant $(x_1y_2 - x_2y_1)^2$ which corresponds to the identical transformation and may be denoted by the symbol E; and we would have a theory of reduction of the symmetric double-quadratic closely analogous to the invariant theory of two ternary quadratics. Without however going into this theory, we shall merely adopt the results already proved for the general double-quadratic. Thus the normal form of $F(xy)$ is

$$a P_x P_y + b Q_x Q_y + c R_x R_y$$

where P, Q, R are the group-quadratics written as before with unit discriminants. $a^2 P^2 + b^2 Q^2 + c^2 R^2$ is the branch quartic and $a P^2 + b Q^2 + c R^2$ (obtained by putting $y = x$ in F) is the *latent quartic* of F. Thus the branch and latent quartics of F belong to its standard pencil. Also

there exists a unique transformation F which has two members of a standard pencil for its branch and latent quartic respectively.

To prove this we have only to express the two members in the forms $a^2 P^2 + b^2 Q^2 + c^2 R^2$ and $a P^2 + b Q^2 + c R^2$, on doing which, we find two solutions for abc of which only one is relevant, the other corresponding to the identical transformation. It follows from this that when F belongs to a family (F) with fixed branch-points, the parameter of the latent quartic of F may be used as a parameter of F itself. Hence, for instance the theorem of § 5 would become :—

If $\lambda_1 \lambda_2 \lambda_3 \lambda\, \mu$ be the parameters of the perfect squares and the branch and latent quartics of F in its standard pencil, then when F is sub-rational μ is a root of the sextic covariant of $(x - \lambda)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$.

Certain properties of unsymmetric transformations have to be slightly modified when the transformations are symmetric. Thus the four homographies of a family with given group-quadratics will now

reduce to the identical transformation E and the three group involutions. The derived transformation corresponding to the relevant quadratic and the λ-transformation of a sub-rational tranformation will now be not merely homographies but involutions,* and so on.

The following identities between the group quadratics P, Q, R will be found useful :—

(1) $P^2 + Q^2 + R^2 = 0.$

(2) $P_x P_y = P^2_{xy} + \frac{1}{2}(x_1 y_2 - x_2 y_1)^2$ where P_{xy} is the polar form of P_x.

(3) $-2\Sigma P^2_{xy} = \Sigma P_x P_y = (x_1 y_2 - x_2 y_1)^2$

It follows that $F(xy)$ can also be written in the form

$$a' P^2_{xy} + b' Q^2_{xy} + c' R^2_{xy}.$$

From these identities one may easily prove the following statements concerning $a P_x P_y + b Q_x Q_y + c R_x R_y$ the normal form of $F(xy)$:—

(1) If two of the co-efficients a, b, c are equal, F is the product of two unsymmetric bilinear forms.

(2) If two of the co-efficients differ in sign only, F splits into symmetric factors.

(3) If $a = b = c$ F is a multiple of $(x_1 y_2 - x_2 y_1)^2$.

(4) If $a^2 = b^2 = c^2$ and a, b, c are not all equal, F is the square of a symmetric form.

(5) If one of the co-efficients a, b, c is zero, F is sub rational.

Permutable Symmetric Transformations.

8. If F_1, F_2 be two symmetric transformations, the transformation defined by $\phi(xy)$, the z-resultant of $F_1(xz)$ and $F_2(zy)$ may be called their *operative product* and denoted by the symbol $F_1 \otimes F_2$. The transformations F_1, F_2 will be said to be permutable if

$$F_1 \otimes F_2 = F_2 \otimes F_1, \text{ i.e. if } \phi(xy) \text{ is symmetric;}$$

* Here and hereafter we shall use 'involution' to denote both a linear involutric transformation and the (2,2) correspondence determined by the square of a symmetric bilinear form.

for instance F is always permutable with its group-involutions and with the identical transformation E.

THEOREM I: *Two transformations with the same group are permutable if and only if they have the same branch-points.*

Taking the group-quadratics to be P, Q, R we may write

$$F_1(xz) = a\,P_xP_z + b\,Q_xQ_z + c\,R_xR_z$$
$$F_2(zy) = a'\,P_zP_y + b'\,Q_zQ_y + c'\,R_zR_y.$$

The z-eliminant $\phi(xy)$ is then

$$(a^2P_x^2 + b^2Q_x^2 + c^2R_x^2)(a'^2P_y^2 + b'^2Q_y^2 + c'^2R_y^2)$$
$$-(aa'\,P_xP_y + bb'\,Q_xQ_y + cc'\,R_xR_y)^2.$$

If $\phi(xy)$ is to be symmetric, the first term of this expression should be symmetric. The necessary and sufficient condition for this is that the equations

$$a^2P_x^2 + b^2Q_x^2 + c^2R_x^2 = 0$$
$$a'^2P_y^2 + b'^2Q_y^2 + c'^2R_y^2 = 0$$

shoud have the same roots, *i.e.*, F_1, F_2 should have the same branch-points.

The permutability of F with its group-involutions is a special case of this theorem, since the group invo utions are extreme members of the family of transformations having the same branch-points as F.

THEOREM 2: *The operative product of two transformations having the same branch-points is also the algebraical product of two other transformations having the same branch-points.*

We have to show that when F_1 and F_2 have the same branch points, $\phi(xy)$ (which is equal to $F_1 \otimes F_2$) $= \phi_1\phi_2$ where ϕ_1 and ϕ_2 are symmetric double-quadratics having the same branch points as F_1, F_2.

Since F_1F_2 have the same branch-points, we have

$$a'^2 - a^2 = b'^2 - b^2 = c'^2 - c^2 = t \text{ (say)}.$$

Using these equations and the identity $P^2 + Q^2 + R^2 = 0$,

$$\phi(xy) = \Sigma\,(bc'\,Q_xR_y - b'c\,Q_yR_x)^2$$

reduces to

$$P_x^2P_y^2\,(b^2c^2 - c^2a^2 - a^2b^2 - a^2t) - 2bb'cc'\,Q_xQ_y\,R_xR_y$$
$$+ \text{ two similar terms.}$$

Thus $\phi(xy)$ is a quadratic in P_xP_y, Q_xQ_y and R_xR_y; and its discriminant is easily seen to be a rational function of t, a, b, c only, of the third degree in t. Hence the discriminant will vanish identically if it vanishes for four values of t. But it is obvious that

$$F_1 \otimes E = F_1 \times F_1,$$
$$F_1 \otimes I_r = F_{1r} \times F_{1r}$$

where E is the identical tranrformation, I_1, I_2, I_3 the group-involutions and

$$F_{1r}(xy) = F_1 [x, I_r(y)] \qquad (r = 1, 2, 3).$$

Thus the discriminant vanishes for four cases in which F_2 has the same branch-points as F_1 and therefore identically. Thus

$$\phi(xy) = \phi_1 \times \phi_2$$

where ϕ_1, ϕ_2 are linear combinations of P_xP_y, Q_xQ_y, R_xR_y, *i.e.* transformations having the same group as F_1 F_2. To show that $\phi_1\phi_2$ have the same branch-points as F_1 F_2, suppose that α is a branch-point of F_1 and not a branch-point of ϕ_1, let F_1 carry α into $(\beta\beta)$ and F_2 carry β into $(\beta_1\beta_2)$. Since α is not a branch-point of ϕ_1, ϕ_1, and ϕ_2 must each carry α into $(\beta_1\beta_2)$. Since F_1 and ϕ_1 have the same group and *one* of the branch-points of F_1 is not a branch-point of ϕ_1, it follows that F_1 and ϕ_1 must have all their branch-points different. Hence ϕ_1 and ϕ_2 will have $4 \times 2 = 8$ corresponding pairs in common and will therefore be identical. Obviously this cannot happen in general (as a simple instance $F \otimes F$ is not usually a perfect square). Hence F_1F_2 ϕ_1 ϕ_2 must all have the same branch-points.

This theorem (which may be called the '*product theorem*') is fundamental in the theory of symmetric transformations and will be quoted in the form $F_1 \otimes F_2 = \phi_1 \times \phi_2$.

One might easily convince oneself that there exist four and only four pairs of transformations F_1,F_2 having a given operative product $\phi_1 \times \phi_2$. These pairs are easily located. Thus if F_1F_2 be one pair, the others are $\qquad F_{1r}, F_{2r} \qquad (r = 1, 2, 3)$

where $\quad F_{1r}(xy) = F_1 [x, I_r(y)]$, $F_{2r}(xy) = F_2 [x, I_r(y).]$

To state this in a more instructive form, let the family of transformations with common branch-points be represented parametrically λ, λ_1, λ_2, λ_3 being the parameters of identity and the group-in-

volutions. Let $i_1\, i_2\, i_3$ be the involutions which carry the quartette $(\lambda\, \lambda_1 \lambda_2 \lambda_3)$ into itself. Then

$$(\alpha\beta), \ [i_r(\alpha), \ i_r(\beta)] \ (r = 1, 2, 3)$$

are four pairs of transformations which have the same operative product. As a particular case

$$F \times F = F \otimes E = F_1 \otimes I_1 = F_2 \otimes I_2 = F_3 \otimes I_3$$

where $\qquad F_r(xy) = F\,[x, I_r(y)]$.

9. The Powers of F.

Since F is symmetric, E the identical transformation must be a factor of the operative product $F \otimes F$. Hence

$$F \otimes F = E \times F' \text{ (say)}.$$

F' will be written F^2 and called the *square of F*. Again it is obvious that $F \otimes F^2 = F \times F''$. F'' will be written F^3 and called the *cube of F*. In the same way all powers of F can be defined by means of equations of the type

$$F \otimes F^m = F^{m-1} \times F^{m+1}.$$

It is important to notice that all powers of F thus defined have the same branch-points as F. The following relations obviously hold between the powers:

$$(F^m)_n = F^{mn}, \quad F^m \otimes F^n = F^{m+n} \times F^{m-n} \qquad (m > n).$$

F^0 may be regarded as merely E and the symmetry of F leads us to define negative powers by the equality $F^{-n} = F^n$.

The results of § 8 enable us to find the relation between the transformations which have the same square. Thus if F^2 is known, the transformations F which satisfy the equation

$$F \otimes F = E \times F^2$$

are of the form F, F_1, F_2, F_3 where $F_r(xy) = F\,[x, I_r(y)]$.

The transformations $F_1 F_2 F_3$ may be called the co-square transformations of F.

THEOREM: *The λ-transformation of F is linearly transformable into F^2.*

For the absolute invariant of the branch-elements of ψ the λ-transformation of F is equal to that of the branch-points of F. Further to any value of ψ there correspond four values of F of the form

F, F_1, F_2, F_3 and therefore a single value of F^2, and *vice versa*. Since this correspondence is such that whenever ψ is identity or an involution, the same is the case with F^2, it follows that ψ is linearly transformable into F^2. The existence of the derived transformations of F and the possibility of exhibiting the λ-transformation of F as a derived transformation of a derived transformation indicate the fact that the parameter of F^2 is a rational function $R_4(t)$ of the parameter of F, of such a kind that $R_4(t)$ can be written in the form $R[S(t)]$ *in three essentially distinct ways*, where R, S are rational quadratic functions.

Ex. (4). There are six sub-rational transformations with given branch-points—two with each group-quadratic as the relevant quadratic. If these be $F_1F'_1$, $F_2F'_2$, F_3F_3', establish the following equations :—

$$F_2 \otimes F_3 = F_2 \otimes F'_3 = F'_2 \otimes F_3 = F'_2 \otimes F'_3 = F_1 \times F'_1$$
$$F_1 \otimes F'_1 = I_2 \times I_3$$
$$F_1 \otimes I_1 = F_1 \times F_1 \ ,F_1 \otimes I_2 = F_1 \otimes I_3 = F'_1 \times F'_1$$
$$F_1^2 = F_1'^2 = I_1.$$

Thus the 10 transformations $E\ I_r\ F_r\ F'_r$ form a closed group with respect to operative and algebraic multiplications.

This is a particular case of a theorem to be noticed later.

Closed Triads of Transformations.

10 Since ϕ_1 is an algebraic factor of the operative product $F_1 \otimes F_2$ in the equation $F_1 \otimes F_2 = \phi_1 \times \phi_2$, it would follow equally that F_1 should be a factor in the operative product $F_2 \otimes \phi_1$, and therefore $F_1 F_2 \phi_1$ form a closed triad of transformations. The search for the relation determining such triads leads us into topics connected with Poncelet's Theorem.

For the sake of clearness, let us adopt from the start a geometrical conception of these transformations. Let the conics of a four-point system be represented by parameters and let the parameters $\omega_1 \omega_2 \omega_3$ belong to the line-conics If any conic e of the system be chosen as the base conic and if PQ be a chord of e touching another given conic of the system, then the relation between P and Q is a symmetric (2,2)

correspondence, the branch-points of which are the common-points of the system. Thus the conics of the four-point system determine on the conic e a family of symmetric transformations with common branch points, the conic e itself determining the identical transformation and the line-conics determining the group-involutions. Obviously three conics of the system touching severally the sides of an inscribed triangle of e, determine a closed triad of transformations of the type $F_1F_2\phi_1$. Let three such conics be said to form *a Poncelet triad* of e. The following statements are obvious :—

(1) If f is any conic of the system, (eff) is a Poncelet triad of e. For the identical transformation E is always a factor of $F \otimes F$.

(2) $(\omega_1\omega_2\omega_3)$ is a Poncelet triad of every conic of the system. For, the group-involutions satisfy the equation $I_2 \otimes I_3 = I_1 \times I_1$.

(3) The equation $I_2 \otimes I_3 = I_1 \times I_1$ also shews that $(\omega_1\omega_2 k)$ is not a Poncelet triad of any conic of the system unless k is ω_3 In other words the normally distinct values of k for which $(\omega_1\omega_2 k)$ is a Poncelet triad coalesce into ω_3, the parameter of the third line-conic, when $\omega_1\omega_2$ are two of the line-conics. From these facts we can arrive at the parametric equation of Poncelet triads. For, represent each triad (lmn) of conics of the system by a point in three-dimensional space (this may be done for instance by taking the Cartesian co-ordinates of the representative point to be Σl, Σmn, lmn). The repeated triads (kkk) are then represented by points k on a certain twisted cubic Γ. Let $\omega_1\omega_2\omega_3$ on Γ correspond to the repeated line-conics and let the osculating planes at $\omega_1\omega_2\omega_3$ intersect in V which therefore represents the the triad $(\omega_1\omega_2\omega_3)$. The locus of the representative points of triads of the form (eff) is evidently the conic of intersection of the osculating plane at e with the tangents of Γ. Denote this conic by S_e.

Now the locus of the representative points of Poncelet triads of e or briefly the Poncelet locus of e is a quadric, which by (1) and (2) above contains the conic S_e and the point V. Further by (3) the three lines of intersection two by two of the osculating planes at $\omega_1\omega_2\omega_3$ are tangents at V to this quadric. But these lines are evidently not co-planar. We have therefore the result:

The Poncelet locus of e *is the cone whose vertex is the representative point of the triad of line-conics and base, the conic S_e.*

Since any point on a cone is a linear combination of the vertex and a point on the base, we see that the parameters of a Poncelet triad are the roots of an equation of the form

$$\lambda f(x) + (x - e)(x - \mu)^2 = 0$$

where $$f(x) = (x - \omega_1)(x - \omega_2)(x - \omega_3)$$

and λ, μ are parameters.*

The same equation represents the closed triads of transformations in a family with fixed branch-points, if we take the parameters of identity and the group-involutions to be e and the roots of $f(x)$ respectively.

The result reached may be rendered picturesque by a plane geometric representation which is of some importance in the theory. Let the conics of a four-point system be made to correspond to points on a conic S the line-conics and the base-conic corresponding to the points ABCE on S. Denote by [E] the system of conics which pass through E and are inscribed in the triangle ABC. *Then the Poncelet triads of E are the conics corresponding to the vertices of triangles which are inscribed in S and circumscribed to the conics [E].*

In particular, the conics of the system to which triangles can be circumscribed so as to be inscribed in E, correspond to the contact-points of the four-conics [E] which touch S; and the conics of the system to which triangles can be inscribed so as to be circumscribed to E correspond to the two points of intersection of S with that conic [E] which touches S at E.

The Square Roots of F.

11. It was shewn that ψ the λ-transformation of F is linearly transformable into F^2. If P, Q, R are the group-quadratics and if ψ is an involution—say the involution corresponding to the division $\delta_x P^2$, Q^2R^2, of the branch-elements of ψ—then it obviously follows that the four values of F coalesce two and two into the two sub-rational transformations with P as the relevant quadratic. Hence also the four square roots of a group-involution coalesce into the two sub-rational transformations with the corresponding group-quadratics as relevant quadratics. Hence if F belongs to a family with fixed branch-points and if $\lambda\lambda_1\lambda_2\lambda_3$ be the parameters of identity and the group-involutions, then the parameter

* The usual invariant form of Poncelet's Theorem (Casey's *Analytical Geometry*) may be easily derived from this equation.

of F^2 is a rational function $R(x) = \dfrac{f(x)}{\phi(x)}$ of the parameter x of F, of such a kind that

$$f(x) - \lambda_1 \phi(x),$$
$$f(x) - \lambda_2 \phi(x),$$
$$f(x) - \lambda_3 \phi(x)$$

are all squares of the corresponding quadratic factors of the sextic co-variant of

$$(x - \lambda)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

This condition suffices for the complete determination of $R(x)$. As an example, let $\lambda_1 \lambda_2 \lambda_3$ be $1, \omega, \omega^2$ (where $\omega^3 = 1$) and let

$$H = -\tfrac{1}{8}(\lambda^2 x^4 + 8x^3 - 18\lambda x^2 + 8\lambda^2 x + 1)$$

the Hessian of $(x - \lambda)(x^3 - 1)$.

It is easily shewn that

$$(1 - \lambda^2\omega)(x^3 - 1)(x - \lambda) + 2(\lambda - \omega)H$$

is the square of the quadratic which is apolar to $(x - \lambda)(x - \omega)$ and $(x - 1)(x - \omega^2)$. Thus the function $R(x)$ is given by

$$R(x) = \frac{(x^3 - 1)(x - \lambda) + 2\lambda H_x}{\lambda^2 (x^3 - 1)(x - \lambda) + 2H_x}.$$

We may use this result to find the transformations of period 3, that is those which satisfy $F^3 = E$. These are the roots of $R(x) = x$ and are therefore given by

$$P_x \equiv \lambda^3 x^4 - 4x^3 + 6\lambda x^2 - 4\lambda^2 x + 1 = 0.$$

It is evident on inspection that P_x is a self-apolar quartic apolar to the cubic $x^3 - 1$ and also to the cubic $x^3 - \lambda x$.

Ex. (5) Let $p_1 p_2$, $q_1 q_2$, $r_1 r_2$ be the pairs which separate harmonically $(\lambda\lambda_1, \lambda_2\lambda_3)$, $(\lambda\lambda_2, \lambda_1\lambda_3)$ and $(\lambda\lambda_3, \lambda_1\lambda_2)$ and let $S(x)$, $T(x)$ be rational quadratic functions defined by the following conditions:

(i) $S(x) = T(x) = \lambda$ when $x = \lambda$ or λ_1,

(ii) $S(x) = \lambda_1$ when $x = \lambda_2$ or λ_3,

(iii) $S(x) = p_1$ ... $x = q_1$ or q_2,

(iv) x has the repeated values p_1, p_2 when $T(x) = \lambda_2$ or λ_3 respectively.

By using the fact that the λ-transformation is a derived transformation of a derived transformation, shew that

$$R(x) = T[S(x)] \text{ identically.}$$

Ex. (6) If $f(x)$ represent parametrically the line-pairs and $\phi(x)$ any other triad of conics of a four-point system, shew that $\phi(x)$ is a Poncelet triad of exactly four conics of the system—the parameters of these conics being the non-repeated roots of those members of the pencil $\lambda f(x) + \phi(x)$ which have repeated roots.

Ex. (7) Deduce from the last exercise and the properties of P_x that if triangles could be inscribed in S so as to be circumscribed to S′ then triangles could be inscribed in S′ with their sides touching severally the conic S and the two harmonic conics of the system $S + kS'$ (*i.e.* the two conics each of which has triangles inscribed in itself circumscribed to the other).

Hence when S′ is given, the two values of S correspond to the factors of the operative product $F_1 \otimes F_2$ where F_1, F_2 are the transformations determined by the harmonic conics on the base conic S′

A Geometrical Transformation corresponding to the Product Theorem.

12. For the further study of the product-theorem, a geometrical method already indicated, may be employed.

Let the family of symmetric transformations with given branch-points be represented as points on a conic S, the points ABCE on S corresponding to the group-involutions and identity. Let XYZ, the harmonic triangle of ABCE be chosen as the triangle of reference and let E be (1, 1, 1) so that ABC are $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ respectively Let the tangential equation of S be

$$aL^2 + bM^2 + cN^2 = 0,$$

where $$bc + ca + ab = 0.$$

Let our letters whenever they stand for points on S, stand also for the transformations which they represent and *vice versa*.

Then the equation $F_1 \otimes F_2 = \phi_1 \times \phi_2$ represents a rational line-transformation which carries the line $\phi_1\phi_2$ into the line F_1F_2. Since $E \otimes F = F \times F$, it follows that any chord of S through E is carried by this transformation into the tangent at its other extremity; so that the point E is carried into the conic S. This shews that the transformation is *quadratic*. Further, it has been shewn that the four values of F_1, F_2 satisfying the equation $F_1 \otimes F_2 = \phi_1 \times \phi_2$ are of the form

F, i_1F, i_2F, i_3F where $i_1 i_2 i_3$ are the involutions which carry the quartette composed of identity and the group-involutions into itself. This shews that L′M′N′ the co-ordinates of the transformed line are functions solely of $L^2 M^2 N^2$. Adding to this the conditions that the sides of xyz are transformed into those of ABC [*Ex.* (4) Page 184] and that any tangent to S is transformed into a line through E we readily deduce that the transformation of lines determined by the equation $F_1 \otimes F_2 = \phi_1 \times \phi_2$ is of the form

$$L' = bM^2 + cN^2$$
$$M' = cN^2 + aL^2$$
$$N' = aL^2 + bM^2.$$

The 7 fixed lines of this transformation comprise the tangent at E and the joins of the 6 associated pairs of transformations of period 5. Thus if $\phi^5 = E$, so that ϕ is of period 5, then ϕ^2 is also of period 5 and $\phi = \phi^4$, $\phi^2 = \phi^5$, so that

$$\phi \otimes \phi^2 = \phi \times \phi^3 = \phi \times \phi^2.$$

Ex. (8). E is a conic of a given 4-point system and E_1, E_2 the two conics of the system which are inscribed to self-polar triangles of E. A triangle is inscribed in E with two of its sides touching E_1, E_2 respectively. Shew that the envelope of the third side is composed of the harmonic conics of the system.

(For, the line E_1E_2 is the polar line of E with respect to ABC. This polar line is (1, 1, 1) and is carried into $(b + c, c + a, a + b)$ which is the polar line of the triangle ABC with respect to S).

It follows that if F_1, F_2 are the harmonic conics,

$$E_1 \otimes E_2 = F_1 \times F_2, \text{ (E being the base conic).}$$

Ex. (7) gave a geometric specification of the algebraic factors of the operative product $F_1 \otimes F_2$. The present exercise gives a similar specification of the operative factors of the algebraic product $F_1 \times F_2$.

A Consequence of the Product Theorem.

13. We shall now deduce from the Product Theorem, a general relation between the transformations F_1F_2 $\phi_1\phi_2$ which is of great importance. It will be well to remember in what follows that the symbol $\otimes$ being merely a symbol of elimination is distributive with respect to $\times$.

Lemma. If $F_1 \otimes F_2 = \phi_1 \times \phi_2$,

then $\phi_1 \otimes \phi_2 = F_1^2 \times F_2^2$

and $F_1^2 \otimes F_2^2 = \phi_1^2 \times \phi_2^2$.

Proof. Since $F_1 \otimes F_2 = \phi_1 \times \phi_2$ we must have

$$F_1 \otimes F_2 \otimes F_1 \otimes F_2 = (\phi_1 \times \phi_2) \otimes (\phi_1 \times \phi_2)$$

that is $$E \times E \times (F_1^2 \otimes F_2^2) \times F_1^2 \times F_2^2 \times F_1^2 \times F_2^2$$
$$= E \times E \times (\phi_1^2 \times \phi_2^2) \times (\phi_1 \otimes \phi_2) \times (\phi_1 \otimes \phi_2).$$

Hence
$$(F_1^2 \otimes F_2^2) \times (F_1^2 \times F_2^2) \times (F_1^2 \times F_2^2)$$
$$= (\phi_1^2 \times \phi_2^2) \times (\phi_1 \otimes \phi_2) \times (\phi_1 \otimes \phi_2).$$

Since this is an identity we can equate the squared and the non-squared factors in the two sides. Hence

$$F_1^2 \otimes F_2^2 = \phi_1^2 \times \phi_2^2 \text{ and } \phi_1 \otimes \phi_2 = F_1^2 \times F_2^2.$$

THEOREM: *If* $F_1 \otimes F_2 = \phi_1 \times \phi_2$, *then for all integral values of* m, n

$$(F_1^{m+n} \otimes F_2^{m-n}) \times (F_1^{m-n} \otimes F_2^{m+n})$$
$$= (\phi_1^m \otimes \phi_2^n) \times (\phi_1^n \otimes \phi_2^m).$$

Proof. Denote the left side of this equation by $[m, n]$ and the right side by $\{m, n\}$. We notice that $[m, n] = [n, m]$ and $\{m, n\} = \{n\ m\}$. Assuming the theorem to be true for values of m, n up to and including r we shall prove it true for values including $r + 1$.

Remembering the distributive character of $\otimes$ with respect to $\times$, we easily find the following identities:

(i) $$[m, n] \otimes (F_1^k \otimes F_2^k) = [m + k, n] \times [m, n + k]$$
$$\times [m - k, \;] \times [m, n - k]$$

(ii) $$\{m, n\} \otimes (\phi_1^k \times \phi_2^k) = \{m + k, n\} \times \{m, n + k\}$$
$$\times \{m - k, n\} \times \{m, n - k\}$$

Now by our assumption the equation

$$[r, n] = \{r, n\} \qquad \dots \qquad \dots \quad (1)$$

is true for values of n up to and including r. Operating on its two sides with $F_1 \otimes F_2$ and $\phi_1 \times \phi_2$ we obtain

$$[r + 1, n] \times [r, n + 1] \times [r - 1, n] \times [r, n - 1]$$
$$= \text{the same with flower brackets} \quad \dots \quad (2)$$

When $n < r$ the last three factors of the two sides of this equation cancel in virtue of (1) and we have

$$[r + 1, n] = \{r + 1, n\} \qquad (n < r) \quad \dots \quad (3)$$

Since (1) is true when $n = r$, we can put $n = r$ in (2). We then obtain

$$[r + 1, r] = \{r + 1, r\} \quad \dots \quad \dots \quad (4)$$

Hence equation (1) and therefore equation (2) is true when $n = r+1$.

Putting $n = r + 1$ in (2) and using (3) we have

$$[r+1, r+1] \times [r, r+2] = \{r+1, r+1\} \times \{r, r+2\} \quad \dots \quad (5)$$

Now, by the lemma $F_1^2 \otimes F_2^2 = \phi_1^2 \times \phi_2^2$. Operating on the sides of (1) with $F_1^2 \otimes F_2^2$ and $\phi_1^2 \times \phi_2^2$ and putting $n = r$, we have

$$[r, r + 2] = \{r, r + 2\}.$$

Substituting in (5) we obtain

$$[r + 1, r + 1] = \{r + 1, r + 1\}. \quad \dots \quad (6)$$

Thus from (3), (4) and (6), the theorem is true for values of m, n including $r + 1$. Since the lemma shews the truth of the theorem for $r = 1$, it follows by induction that the theorem is true universally.

Cor. (1) Putting $n = 0$, we have the important result:

If $F_1 \otimes F_2 = \phi_1 \times \phi_2$ *then* $F_1^m \otimes F_2^m = \phi_1^m \times \phi_2^m$.

In other words, any the same powers of a closed triad of transformation also form a closed triad.

Cor. (2) One of the factors of the operative product of two transformations having the same mth power is a mth root of identity.

Thus if $F_1^m = F_2^m = S$ and $F_1 \otimes F_2 = \phi_1 \times \phi_2$,

then $\phi_1^m \times \phi_2^m = F_1^m \otimes F_1^m = S \otimes S = E \times S^2$.

Cor. (3) The result of Cor. (1) may be extended. Thus it is easy to shew by induction that if the continued operative product

$$F_1 \otimes F_2 \otimes F_3 \otimes \dots = \phi_1 \times \phi_2 \times \phi_3 \dots$$

then

$$F_1^m \otimes F_2^m \otimes F_3^m \otimes \dots = \phi_1^m \times \phi_2^m \times \quad \dots$$

Cor. (4) The operative product of two roots of identity is also the algebraic product of two other roots of identity.

Thus if $F_1^{k_1} = F_2^{k_2} = E$

and $F_1 \otimes F_2 = \phi_1 \times \phi_2$

then $\phi_1^k = \phi_2^k = E$,

where k is the least common multiple of k_1, k_2.

Hence the nth roots of identity form a closed set for operative and algebraic multiplications. [Cf. Ex. (4)].

Ex. (9) If E be a conic of a given four-point system, E_1, E_2 the two conics of the system which are inscribed to self-polar triangles of E and S_1, S_2 the two conics of the system which are circumscribed to circumscribed triangles of E, then if a triangle be inscribed in E with two of its sides both touching one of the conics E_1, E_2, the envelope of its third side will be one of the conics S_1, S_2.

This may be deduced from Ex. (7) and Ex. (8) by means of the Lemma.

Cyclic Transformations.

17. If $F^m = E$, F may be called a *cyclic* or periodic transformation, and m may be said to be *a period* or *the period* of F according as m is not or is the least number for which $F^m = E$. The simpler types of transformations already met with are cyclic; thus the involution represented by the square of the symmetric bilinear form is the transformation of period two and the sub-rational transformation is the transformation of period four. In the conic-representation of symmetric transformations, a cyclic transformation of period m corresponds to a conic which is inscribed in a m-gon inscribed in the base-conic.

If $F^m = E$ we will also say that F is a mth root of identity; if in addition no power of F lower than the mth is E we will say that F is a *special* mth root of identity.

Number of the distinct and the special m*th* roots of identity.

A transformation F has four square roots and therefore m^2 m-th roots whenever m is a power of 2. If generally $f(m)$ is the number of mth roots, then since all the mnth roots are all the nth roots of all the mth roots, we must have

$$f(mn) = f(m)\,f(n).$$

Hence $f(m)$ must be a power of m; and by comparison with the case $m = 2^k$ we see that $f(m)$ is always $= m^2$. When F is a general transformation, its m_2 mth roots are all distinct; but when F is identity the mth roots coalesce in a special manner. To find the number of distinct mth roots of identity we must have recourse to the results of § 13. Let $f_1 f_2 \ldots f_{m^2}$ be the distinct m^2 mth roots of a non-cyclic transformation f and $\omega_1 \omega_2 \ldots \omega_\pi$ be the distinct mth roots of identity. Then by cor. (1) § 13 we have π equations of the type

$$f_1 \otimes \omega_t = f_p \times f_q \qquad (t = 1, 2 \ldots \pi).$$

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NOTES AND QUESTIONS.

Notes and Questions.

The Perpetual Calendar Formula.*

1. The Calendar Problem is to find the day of the week on any date of the Christian year. Several charts have been prepared to serve this purpose. One of these by the author of this paper was published in this *Journal* in October 1923. The aim of this paper is to arrive at a *formula* that will replace the long process adopted at present in solving Calendar Problems, so that it will then be a matter of substitution merely.

2. *Definitions.* If k and n be two positive integers we shall denote by $Q\left(\frac{k}{n}\right)$ the quotient k_1 obtained by dividing k by n; and by $R\left(\frac{k}{n}\right)$ the remainder k_2 so obtained. This will also be denoted by writing $k \equiv k_2 \pmod{n}$.

If we have two positive integers k, l such that

$$k = k_1 n + k_2$$

$$l = l_1 n + l_2 \qquad [0 \leqslant k_2, l_2 < n$$

then

$$\frac{k+l}{n} = k_1 + l_1 + \frac{k_2 + l_2}{n}$$

whence

$$Q\left(\frac{k+l}{n}\right) = Q\left(\frac{k}{n}\right) + Q\left(\frac{l}{n}\right) + \lambda$$

where λ is either 0 or 1.

Hence if the terms of an Arithmetic Progression

$$l + k,\ l + 2k,\ l + 3k\ \ldots\ \ldots$$

be divided by n, the successive quotients form a sequence

$$a_1, \quad a_2, \quad a_3, \quad \ldots\ \ldots$$

each term of which exceeds the one on its left by either d or $d + 1$ where

$$d = Q\left(\frac{k}{n}\right).$$

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* Read at the Fifth Conference of the Indian Mathematical Society, Bangalore, 1926.

3. **We now proceed to establish under certain limitations the Converse Theorem,** *viz.*

Given a set of n numbers $a_1, a_2, a_3, \dots a_n$ such that $a_r - a_{r-1} = d$ or $d + 1$, the a's being arranged in ascending order of magnitude, the numbers of the set may be obtained by giving to x in order the values 1, 2, 3, ... , n in an expression of the form $Q\left(\frac{kx+l}{n}\right)$ k, and l being determinate constants.

If possible, let

$$a_r = Q\left(\frac{kr+l}{n}\right) \qquad [r = 1, 2 \dots n$$

so that

$$kr + l = a_r n + b_r \qquad 0 \leqslant b_r < n.$$

Replacing r by $r-1$ and subtracting we have

$$k = (a_r - a_{r-1})\,n + (b_r - b_{r-1}).$$

Now since $d = Q\,(k/n)$, k must lie between nd and $(n+1)\,d$ and hence if

$$a_r - a_{r-1} = d, \quad b_r > b_{r-1} \text{ and } b_r - b_{r-1} = (k - nd) = c \text{ (say)}$$

while

if $$a_r - a_{r-1} = d + 1, \quad b_r < b_{r-1}$$

and $$b_r - b_{r-1} = k - nd - n = c - n.$$

The numbers from 2 to n are thus divided into two groups A and B, the former consisting of those values of r for which $a_r - a_{r-1} = d$, and the latter of those for which $a_r - a_{r-1} = d + 1$. From what has been established, we can show that if p of the numbers 2, 3, ... q belong to A, and the remaining $(q - p - 1)$ to B, then

$$b_q - b_1 = (q-1)\,c - (q-p-1)\,n,$$

or b_1 and b_q being each less than n,

$$\frac{(q-p)\,n}{q-1} > c > \frac{(q-p-2)\,n}{q-1}.$$

If we can find a value of c, satisfying these inequalities for different values of q, each b is known in terms of b_1; b_1 will then be selected so as not to make any $b \geqslant n$. k will thus be known and l will then be equal to $a_1 n + b_1 - k$ with the proper sign.

Example. Take the set of numbers

2, 5, 7, 10, 12, 15, 18, 20, 23, 25, 28, 31.

Here $d = 2$; the numbers 3, 5, 8, 10 belong to Group A and the numbers 2, 4, 6, 7, 9, 11, 12, belong to Group B. The pairs of values of q and p are :—

$$(2, 0);\ (3, 1);\ (4, 1);\ (5, 2);\ (6, 2);\ (7, 2);\ (8, 3);\ (9, 3);\ (10, 4);\ (11, 4);\ (12, 4),$$

considering the limits for c, we obtain $c = 7$.

Hence $k = dn + c = 31$.

The b's in terms of b_1 are

$$b_1, b_1 - 5, b_1 + 2, b_1 - 3, b_1 + 4, b_1 - 1, b_1 - 6, b_1 + 1, b_1 - 4, b_1 + 3, b_1 - 2, b_1 - 7.$$

Since each of these must lie between 0 and 12 we see that b_1 must be 7. This gives $l = 0$. Hence the given set of numbers are given by the expression

$$Q\left(\frac{31x}{12}\right), \qquad [x = 1, 2, 3, \ldots\ldots 12.$$

4. Now every fourth year of a century is a leap year. In the case of complete centuries the year in order to be leap, has to be divisible by 400. In every 400 years we have 97 leap years. Also we make use of the fact that the 1st January in 1 A.D. was a Monday. The calculation now consists in counting the extra days.

It is easily shown that 1, 2, 3, 4, 5, ... complete centuries give in order 5, 3, 1, 0, 5, 3, 1, 0, ... extra days. If c be the number of complete centuries, the number of extra days is given by $5\,R\left(\frac{c}{4}\right)$. The number of extra days for y complete years of the new century is easily seen to be given by $Q\left(\frac{5y}{4}\right)$.

For the purpose of this paper, we shall regard January and February as the 13th and the 14th months of the preceding year. With this convention we shall write 11th February 1913, as 11-14-1912. But 2nd October 1869 will be written as usual as 2-10-1869.

The question being put in this form, the date number, generally denoted by d, is called the d-index; the month number, generally denoted by m, the m-index. The c and y indices are similarly defined. In the former case above we have $d = 11$, $m = 14$, $c = 19$, $y = 12$. In the latter case we have $d = 2$, $m = 10$, $c = 18$ and $y = 69$.

Having considered the number of *extra* days for the c centuries and y years, we have to count

2, 5, 7, 10, 12, 15, 18, 20, 23, 25, 28, 31

extra days according as the month in question is

March, April, May ... January or February.

If m be the m-index the number of these *extra* days is given by

$$Q\left\{\frac{31(m-2)}{12}\right\}.$$

For the d days of the month in question, we have to take d extra days.

The total number of extra days is thus

$$D \equiv d + Q\left[\frac{31(m-2)}{12}\right] + 5R\left(\frac{c}{4}\right) + Q\left(\frac{5y}{4}\right) \qquad (\text{mod. } 7)$$

$$\equiv d+2(m+1)+Q\left[\frac{7m-2}{12}\right] - 2R\left(\frac{c}{4}\right)+y+Q\left(\frac{y}{4}\right).$$

D is called the D-index, it being 0, 1, 2, 3, 4, 5 or 6 according as the Day corresponding to the date in question is Sunday, Monday, Tuesday, ..., Friday or Saturday. In practice, the latter formula is by far the simpler.

Example. Find the day of the week on:

(i) 23rd July 1856. (ii) 13th January 1925

Solution (i) (*a*) 18 complete centuries give 3 extra days.

56 „ years give 70 extra days.

For December subtract 3 days,

„ November „ 2 „ ,

„ October „ 3 „ ,

„ September „ 2 „ ,

„ August „ 3 „ ,

„ July „ 3 „ ,

also add 23 extra days.

Thus we have 80, ≡ 3 (mod. 7) extra days in all.

The day must therefore have been a Wednesday.

(b) By the formula we have

$$D \equiv 23 + 2(7+1) + Q\left(\frac{7.7-2}{12}\right)$$

$$-2R\left(\frac{18}{4}\right) + 56 + Q\left(\frac{56}{4}\right) \pmod{7}.$$

$$\equiv 23 + 16 + 3 - 4 + 56 + 14 \equiv 3 \pmod{7}.$$

(ii) (a) 19 complete centuries give 1 extra day

24 „ years give 30 extra days

For January we have 13 days

Thus we have $44 \equiv 2 \pmod{7}$ extra days.

The 13th Jan. 1925 was therefore a Tuesday.

(b) Here $d = 13$, $m = 13$, $c = 19$, $y = 24$,

$$\therefore D \equiv 13 + Q\left(\frac{31{,}11}{12}\right) - 2R\left(\frac{c}{4}\right) + Q\left(\frac{5.24}{4}\right) \pmod{7}$$

$$\equiv 13 + 28 - 6 + 30 \equiv 2 \pmod{7}.$$

This gives the same result.

HANS R. GUPTA.

Two Well-known Definite Integrals.

The definite integrals of which the standard forms may be taken to be

$$C = \int_0^\infty \frac{y \cos x \, dx}{x^2 + y^2}, \quad S = \int_0^\infty \frac{x \sin x \, dx}{x^2 + y^2},$$

y being a positive constant, are discussed in most text-books and evaluated by methods, which, if they do not involve contour integration, are artificial and (frequently) faulty. The method given below is too simple to be new, though I have not seen it used anywhere except in evaluating S for the special value $y = 0$. The various integrations are, it will be noticed, of functions of the complex quantity $z = x + iy$, with respect to the real part x only. The integrals are convergent if $y \neq 0$, and the limits may be taken to be very large integral multiples of π.

$$\int_{-\infty}^\infty \frac{\sin z}{z} dx = \underset{p \to \infty}{\text{Lt}} \int_{-p\pi}^{p\pi+\pi} \frac{\sin z}{z} dx$$

$$= \underset{p \to \infty}{\mathrm{Lt}} \sum_{-p}^{p} \int_{n\pi}^{n\pi+\pi} \frac{\sin z}{z}\, dx = \underset{p \to \infty}{\mathrm{Lt}} \int_0^{\pi} \sin z \, . dx \sum_{-p}^{p} \frac{(-1)^n}{z + n\pi}$$

$$= \int_0^{\pi} \sin z \, dx \, . \frac{1}{\sin z} = \int_0^{\pi} dx = \pi. \qquad \ldots \quad (1)$$

Similarly

$$^*\int_{-\infty}^{\infty} \frac{\cos z}{z}\, dx = \int_0^{\pi} \cot z \, dx = \underset{p \to \infty}{\mathrm{Lt}} \int_0^{\pi} dx \sum_{-p}^{p} \frac{1}{z + n\pi} = \int_{-\infty}^{\infty} \frac{dx}{z}$$

$$= \int_{-\infty}^{\infty} \frac{(x - iy)\, dx}{x^2 + y^2} = \int_{-\infty}^{\infty} \frac{- iy \, dx}{x^2 + y^2} = - i\pi \ \ldots \quad (2)$$

To evaluate C and S we proceed as follows :—

Multiply (1) by i and add to (2), and we have

$$0 = \int_{-\infty}^{\infty} \frac{e^{iz} dx}{z} = \int_{-\infty}^{\infty} \frac{(x - iy)\, e^{ix-y}\, dx}{x^2 + y^2}$$

$$= e^{-y} \int_{-\infty}^{\infty} \frac{(x - iy)\,(\cos x + i \sin x)\, dx}{x^2 + y^2}$$

$$= 2\, i e^{-y}\, (\mathrm{S} - \mathrm{C}), \ \ldots \text{ the other terms vanishing.}$$

$$\therefore \quad \mathrm{S} = \mathrm{C}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Multiply (2) by i and add to (1), and we have

$$2\pi = i \int_{-\infty}^{\infty} \frac{e^{-iz}\, dx}{z} = i e^{y} \int_{-\infty}^{\infty} \frac{(x - iy)\,(\cos x - i \sin x)\, dx}{x^2 + y^2}$$

$$= 2 e^{y}\, (\mathrm{S} + \mathrm{C}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

From (3) and (4), we have

$$\mathrm{S} = \mathrm{C} = \frac{\pi}{2} e^{-y}.$$

BALAK RAM.

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* Throughout this paper the integral $\int_{-\infty}^{+\infty}$ is to be taken to mean $\underset{p \to \infty}{\mathrm{Lt}} \int_{-p}^{+p+\alpha}$ where α is either zero or π.

Solutions.

Question 353.

(S. RAMANUJAN) :—If n is any positive odd integer, show that

$$\int_0^\infty \frac{\sin nx \, dx}{(\cosh x + \cos x)\, x} = \frac{\pi}{4},$$

and hence prove that

$$\frac{\pi}{8} = \frac{1}{\cosh \frac{\pi}{2n} + \cos \frac{\pi}{2n}} - \frac{1}{3} \cdot \frac{1}{\cosh \frac{3\pi}{2n} + \cos \frac{3\pi}{2n}} + \dots \dots$$

Remarks by S. D. S. Chowla.

On p. 106, Vol. VIII of this *Journal*, Professor Wilkinson has established the first part of the result by contour integration. The second part may be more directly deduced from the first as follows, instead of by the method given on pp. 108—110.

A familiar result in Cauchy's Theory of Reciprocal Functions is the following:—

If $$\int_0^\infty \phi(x) \sin knx \, dx = \psi(n)$$

then $$\alpha [\phi(\alpha) - \phi(3\alpha) + \phi(5\alpha) - + \dots \dots] = \psi(\beta) - \psi(3\beta) + - \dots \dots \dots \quad (1)$$

with the condition $\alpha\beta = \frac{\pi}{2k}$.*

In particular, if $\phi(x) = \frac{1}{x(\cosh x + \cos x)}$ and $n = 1$,

k being an odd + *ve* integer then since right-hand side of (1) $= \frac{\pi}{4} = \psi(1)$ by the first part of the question, we have, on taking

$$\alpha = \frac{\pi}{2k}, \ \beta = 1.$$

* This can be proved by the method given in Bromwich : *Theory of Infinite Series*, p. 447.

$$\frac{1}{\left(\cosh\frac{\pi}{2k}+\cos\frac{\pi}{2k}\right)}-\frac{1}{3}\frac{1}{\left(\cosh\frac{3\pi}{2k}+\cos\frac{3\pi}{2k}\right)}+\frac{1}{5}\frac{1}{\left(\cosh\frac{5\pi}{2k}+\cos\frac{5\pi}{2k}\right)}-+\ldots$$

$$=\frac{\pi}{4}(1-1+1-1+-+\ldots)$$

$$=\frac{\pi}{4}\cdot\frac{1}{2}=\frac{\pi}{8} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

taking the Cesaro sum of the oscillating series on the right, and using $\psi(1)=\psi(3)=\psi(5)=\psi(7)=\ldots=\frac{\pi}{4}$.

Thus the result is proved (by changing k into n).

2. Similarly the result

$$\int_0^\infty \frac{\sin rx}{x}\,dx=\frac{\pi}{2} \text{ when } r \text{ is } +ve,$$

gives by using (1)

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4},$$

a well-known result.

3. By putting $n=1$ in (2) we get (i) of Q. 626.

Question 724.*

(S. Ramanujan) :—Show that

(i) $\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}$ ... ... to n terms

$$=\tan^{-1}\frac{1}{1+2.1^3}+\tan^{-1}\frac{1}{3+2.3^3}+\tan^{-1}\frac{1}{5+2.5^3}+\ldots \text{ to } n \text{ terms}$$

(ii) $\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+3)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+5)\sqrt{3}}$ + ... to n terms

$$=\tan^{-1}\frac{1}{(\sqrt{3})^3}+\tan^{-1}\frac{1}{(3\sqrt{3})^3}+\tan^{-1}\frac{1}{(5\sqrt{5})^3}+\ldots \text{ to } n \text{ terms.}$$

* Corrected.

Remarks by Mehr Chand Suri.

Both parts of the question are particular cases of the identity

$$\tan^{-1}\frac{x}{2n+1}+\tan^{-1}\frac{x}{2n+3}+\tan^{-1}\frac{x}{2n+5}+\ldots\ldots \text{ to } n \text{ terms}$$

$$=\tan^{-1}\frac{x(x^2+1)}{4.1^3+1(3x^2-1)}+\tan^{-1}\frac{x(x^2+1)}{4.3^3+3(3x^2-1)}$$

$$+\tan^{-1}\frac{x(x^2+1)}{4.5^3+5(3x^2-1)}+\ldots\ldots \text{ to } n \text{ terms}$$

which may be established by Mathematical Induction exactly on the lines of the solution of the question given by Mr. K. B. Madhava on page 191, Vol. VIII (1916).

If we put $x=1$ and $x=\frac{1}{\sqrt{3}}$ in it successively, we get the first and second parts of the question. By giving other values to x, other results may be obtained.

Question 1006.

(K. B. Madhava) :—It is known that the hypergeometric series $F(a, b, c, x)$ diverges if $a+b=c$. Show that as $x \to 1-0$ the series becomes infinite like

$$\frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\log\left(\frac{1}{1-}\right).$$

Solutions by S. R. Ranganathan, Sadanand and V. Thiruvenkatacharya.

According to Cesaro's Theorem* if $\frac{b_n}{a_n}$ approaches a definite limit, finite or infinite,

then $$\lim_{x\to 1}\left[\frac{\Sigma\, b_n x^n}{\Sigma\, a_n x^n}\right]=\lim_{n\to\infty}\frac{b_n}{a_n}.$$

so that, we have

$$\lim_{x\to 1}\frac{F(a,b,a+b,x)}{-\log(1-x)}$$

$$=\underset{x\to 1}{\mathrm{Lt}}\frac{\Sigma\,\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(a+b).x^n/\Gamma(a)\,\Gamma(b)\,\Gamma(n+1)\,\Gamma a+b+n)}{\Sigma\, x^n/n}$$

$$=\frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}.\ \underset{n\to\infty}{\mathrm{Lt}}\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(n)\,\Gamma(a+b+n)}.$$

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* *Cf.* Bromwich : *Theory of Infinite Series*, (1908), page (32).

Substituting the asymptotic value given by

$$\Gamma(y+a) \sim \Gamma(y)\, y^a$$

we have
$$\frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} \cdot \underset{n\to\infty}{\text{Lt}} \frac{[\Gamma(n)]^2 . n^a n^b}{[\Gamma(n)]^2 . n^{a+b}}$$
$$= \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}.$$

Thus, although F (a, b, c, x) and log $(1 - x)$ both diverge when $x \to 1$ and $c = a + b$, the limit found above is finite and determinate. Hence, the inference that the hypergeometric series becomes infinite like

$$\frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} \log\left(\frac{1}{1-x}\right) \text{ as } x \to (1-0).$$

It is stated that x approached unity from the left, since on the right of unity, as well as at unity, the series diverges.

[*Vide* also Whittaker and Watson: *Modern Analysis* (1915), page 293].

Question 1070.

(S. RAMUNAJAN, F.R.S.):—Show that

(i) $$\sqrt{\sqrt[5]{\frac{1}{5}} + \sqrt[5]{\frac{4}{5}}} = \sqrt[5]{1 + \sqrt[5]{2} + \sqrt[5]{8}}$$
$$= \sqrt[5]{\frac{16}{125}} + \sqrt[5]{\frac{8}{125}} + \sqrt[5]{\frac{2}{125}} - \sqrt[5]{\frac{1}{125}}.$$

(ii) $$\sqrt[3]{\sqrt[5]{\frac{32}{5}} - \sqrt[5]{\frac{27}{5}}} = \sqrt[5]{\frac{1}{25}} + \sqrt[5]{\frac{3}{25}} - \sqrt[5]{\frac{9}{25}}.$$

(iii) $$\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}} = \frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}.$$

Solution by S. D. Chowla, N. B. Mitra, and S. V. Venkataraya Sastri.

Let a, b, c, d be numbers satisfying the relations—

$$a^5 = 2b^5 = 8c^5 = 16d^5$$

Then, as

$$b^2 = 2cd, \; a^2 = 2bd, \; bc = ad.$$

it is easily proved by squaring that

$$a + b + c - d = \sqrt{(c^2 + 2ab) + (d^2 + 2ac)} \quad \dots \quad (1)$$

Putting

$$a = \sqrt[5]{\frac{16}{125}},\ b = \sqrt[5]{\frac{8}{125}},\ c = \sqrt[5]{\frac{2}{125}},\ d = \sqrt[5]{\frac{1}{125}}$$

in (1) and simplifying we get the first part of (i).

By actual expansion we have

$$\left\{\sqrt[5]{\frac{1}{5}} + \sqrt[5]{\frac{4}{5}}\right\}^5 = \left\{1 + \sqrt[5]{2} + \sqrt[5]{8}\right\}^2.$$

Thus (i) is completely established.

Next, consider the relations

$$9\alpha^5 = 3\beta^5 = \gamma^5.$$

Then, as

$$3\alpha^2\beta = \gamma^3,\quad \beta^2\gamma = \alpha\gamma^2$$

we get by cubing

$$\alpha + \beta - \gamma = \sqrt[3]{(\alpha^3 + 3\beta\gamma^2) + (\beta^3 - 6\alpha\beta\gamma)}. \qquad \ldots \quad (2)$$

Putting

$$\alpha = \sqrt[5]{\frac{1}{25}},\ \beta = \sqrt[5]{\frac{3}{25}},\ \gamma = \sqrt[5]{\frac{9}{25}}$$

in (2), the required result (ii) follows.

Again

$$\left\{\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\right\}^4 = \frac{6(\sqrt{5}+1) + 4.5^{\frac{1}{4}}(\sqrt{5}+1)}{6(\sqrt{5}+1) - 4.5^{\frac{1}{4}}(\sqrt{5}+1)} = \frac{6 + 4\sqrt[4]{5}}{6 - 4\sqrt[4]{5}}$$

$$= \frac{3 + 2\sqrt[4]{5}}{3 - 2\sqrt[4]{5}},$$

which proves (iii)

Question 1235.

(M. VAIDYANATHAN, M.A.):—Three events A, B, C are known to have happened in the same century. What is the chance that the events happened in the order A B C, B happening within n years of the middle of the century.

Solution by S. Audinarayanan.

The three events A, B, C can occur in 3! or 6 different ways and so the chance of the events occurring in the given order is $\frac{1}{6}$, provided they are independent of one another.

If the events happen in the order A, B, C, we shall find the chance for B occurring within n years of the middle of the century. Let x be the number of years from the beginning of the century to the event B. Then for any assigned value of x the number of triple cases is

$x(100-x)$. Hence the number of favourable cases divided by the whole number, that is the probability,

$$\frac{\int_{50-n}^{50+n} x(100-x)\,dx}{\int_{0}^{100} x(100-x)\,dx} = 3\cdot\frac{n}{100} - 4\left(\frac{n}{100}\right)^3.$$

Hence we have the total probability of the events happening in the order A, B, C; B happening within n years of the middle of the century is

$$\frac{1}{6}\left[\frac{3n}{100} - 4\left(\frac{n}{100}\right)^3\right] = \frac{n}{200} - \frac{2n^3}{3000000}.$$

Question 1310.

(V. V. SATYANARAYANA MURTHI) :—Prove that the axis of a conic inscribed in a fixed quadrilateral whose sides touch a circle envelop a parabola whose directrix is the line through the mid-points of the diagonals of the quadrilateral.

Solution by (1) *A. Narasinga Rao,* (2) *S. Audinarayanan.*

(1) Let $F \equiv al^2 + \ldots 2fmn + \ldots = 0 \ldots$ ... (1)

be the tangential equation of a conic referred to rectangular axes and $(l_1\, m_1\, n_1)$ any line in its plane. The equation of its pole with respect to the conic is

$$l\frac{\partial F}{\partial l_1} + m\frac{\partial F}{\partial m_1} + n\frac{\partial F}{\partial n_1} = 0. \qquad \ldots \quad (2)$$

If $(l_1\, m_1\, n_1)$ be an axis of the conic, this pole is at infinity in a direction perpendicular to that of the line. Hence

$$\frac{\partial F}{\partial n_1} = 0, \qquad \ldots \qquad \ldots \quad (3)$$

and

$$\frac{\partial F}{\partial l_1} : \frac{\partial F}{\partial m_1} = l_1 : m_1. \qquad \ldots \qquad \ldots \quad (4)$$

Now let the conic F belong to a four line system. One conic of this system may be reduced to the form

$$a'l^2 + b'm^2 + c'n^2 = 0$$

by a proper choice of axes, so that we have only to write $a + \lambda a'$, $b + \lambda b'$, and $c + \lambda c'$ in place of a, b, c in (1) so as to obtain the

equation of any conic of the system. Making these substitutions in (3) and (4), we have

$$gl_1 + fm_1 + (c + \lambda c')n_1 = 0, \ldots \qquad \ldots \quad (3')$$

and

$$\frac{(a + \lambda a')\, l_1 + hm_1 + gn_1}{l_1} = \frac{hl_1 + (b + \lambda b')m_1 + fn_1}{m_1} \ldots \quad (4')$$

Substituting the value of λ from (3′) in (4′) and simplifying the required envelope is given by

$$c'n \{ h(m^2 - l^2) + (a - b)\, lm + gmn - fnl \} +$$
$$lm(b' - a')\,(gl + fm + cn) = 0. \qquad \ldots \quad (5)$$

Hence the envelope is, *in general*, a curve of class 3.

Now, any pair of perpendicular diameters may be taken as the axes of a circle. Hence if a system of conics contains a circle, the axes-envelope breaks up into the centre of that circle and a residual curve of class one less. Thus when $a' = b'$ we see from (5) that the envelope breaks into the origin $n = 0$ and the parabola

$$h\,(m^2 - l^2) + (a - b)\, lm + gmn - fnl = 0. \qquad \ldots \quad (6)$$

The two principal axes of any conic of the system meet on the centre locus which is here the line through the mid-points of the diagonals and are perpendicular tangents of the parabola (6). Hence this line is the directrix of the parabola.

If $c' = 0$ so that one conic of the system is a pair of points at infinity, the conics are inscribed in a parallelogram and have their axes parallel. Hence the envelope splits up into a pair of points at infinity, and a residual point evidently the centre of the parallelogram (which is also the centre of the unique circle of the system).

If the system contains two circles, the envelope breaks into the centre of the two circles, and the point at infinity in the direction of their radical axis.

These results may also be deduced from equation (5) by writing

$$h = 0,\ f = 0,\ (a - b)\, c = g^2.$$

(2) Take any conic S inscribed in the quadrilateral. The triangle formed by the diagonal lines is self polar with respect to S, The two principal axes of S and the line at infinity form another triangle also self polar to S. Therefore there exists a conic touching these six lines.

The directrix of this parabola passes through the centre of S since the axes of S are two perpendicular tangents to it; it also passes through the orthocentre of the diagonal triangle. But this point is the

centre of the in-circle of the quadrilateral, since the circle is the polar circle of the triangle. The directrix thus passes through the centres of two of the in-conics of the quadrilateral and is therefote the line of centres of all in-conics or the line of collinearity of the mid-points of the diagonals.

Hence the axes envelope a parabola touching the three diagonals and having for directrix the line through the mid-points of the diagonals.

Question 1362.

(S. NARAYANA IYER) :—Show that

$$\frac{1 - \operatorname{dn} u \operatorname{dn} v \operatorname{dn}(u+v)}{1 - \operatorname{cn} u \operatorname{cn} v \operatorname{cn}(u+v)} = k^2.$$

Solution and Remarks by V. Thiruvenkatacharya, M.A., L.T.

This question can be easily solved either with the help of Liouville's Theorem or by using the formulæ for $\operatorname{dn}(u+v)$ and $\operatorname{cn}(u+v)$. By the latter method, the numerator reduces to

$$k^2 (\operatorname{sn}^2 u + \operatorname{sn}^2 v - \operatorname{sn}^2 u \operatorname{sn}^2 v - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v + \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v \operatorname{dn} u \operatorname{dn} v)$$

and the denominator to

$$\operatorname{sn}^2 u + \operatorname{sn}^2 v - \operatorname{sn}^2 u \operatorname{sn}^2 v - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v + \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v \operatorname{dn} u \operatorname{dn} v.$$

Hence the result.

Proper substitutions for the arguments in this formula lead us to other interesting results showing the relation between $\operatorname{sn}(u+v)$ and $\operatorname{cn}(u+v)$ or $\operatorname{sn}(u+v)$, $\operatorname{cn}(u+v)$ and $\operatorname{dn}(u+v)$, though not quite as elegant.

Changing u into $u + K$ and v into $v + iK'$ in the above question and after making some reductions, we have

$$\frac{\operatorname{sn} u \operatorname{dn} v + \operatorname{dn} u \operatorname{sn} v \operatorname{cn}(u+v)}{\operatorname{cn} v \operatorname{sn}(u+v)} = 1; \quad \dots \quad (1)$$

or changing u into $u + iK'$ and v into $v + K$, we have

$$\frac{\operatorname{sn} v \operatorname{dn} u + \operatorname{dn} v \operatorname{sn} u \operatorname{cn}(u+v)}{\operatorname{cn} u \operatorname{sn}(u+v)} = 1; \quad \dots \quad (2)$$

Similar results but more complicated and less symmetrical can be obtained by changing u into $u+K$ or $u+iK'$; or v into $v+K$ or $v+iK'$.

Questions for Solution.

Corrections:—*Questions 1421 and 1425 which appeared under the name of Mr. M. V. Ramakrishnan were proposed by Mr. B. B. Bagi of Karnatak College, Dharwar.*

Attention is also drawn to Question 1423 which is reprinted after correction.

1423. **(Corrected)**: (V. RAMASWAMI AIYAR) :—ABCD is a spherical quadrilateral whose sides are supposed rigid and jointed at the corners and in which the sums of the opposite sides are equal. If one of the sides be held fixed and the quadrilateral deformed on the sphere, show that the locus of the *incentre* of the quadrilateral is a sphere-conic whose minor axis subtends a right-angle at every point on the curve, the length of this axis being

$$\tan^{-1}\left\{\frac{2}{\sin s}\sqrt{\sin a \cdot \sin b \cdot \sin c \cdot \sin d}\right\},$$

where a, b, c, d are the sides and s is the semi-perimeter of the quadrilateral.

1426. (S. D. CHOWLA) :—If α, β are positive numbers such that $\alpha\beta = \pi^2$, prove that

$$\frac{[\eta(\alpha)]^8}{e^{\alpha}} + \frac{[\eta(\beta)]^8}{e^{\beta}} = \frac{1}{16}$$

where $\eta(x)$ denotes

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n e^{-nx}\left[(1-e^{-x})(1-e^{-3x})\ldots(1-e^{-(2n-1)x})\right]}{(1-e^{-2x})(1-e^{-4x})(1-e^{-6x})\ldots(1-e^{-2nx})}.$$

1427. (S. D. CHOWLA) :—Use Cauchy's Theory of Reciprocal Functions to prove Riemann's famous result

$$2^{1-s}\,\Gamma(s)\,\zeta(s)\cos\left(\tfrac{1}{2}s\pi\right) = \pi^s\,\zeta(1-s).$$

1428. (S. D. CHOWLA) :—Prove that

$$e^{-\beta^2} - 3e^{-9\beta^2} + 5e^{-25\beta^2} - 7e^{-49\beta^2} + - \ldots$$

$$= \frac{\sqrt{2}\,.\,\pi^{\frac{3}{2}}}{8\sqrt{2\beta^3}}\left\{e^{\frac{-\pi^2}{16\beta^2}} - 3e^{\frac{-9\pi^2}{16\beta^2}} + 5e^{\frac{-25\pi^2}{16\beta^2}} - 7e^{\frac{-49\pi^2}{16\beta^2}} + - \ldots\right\}$$

1429. (S. D. CHOWLA):—Prove that

$$\frac{1}{3^4}+\frac{3^4-1}{3^4.5^4}+\frac{(3^4-1)(5^4-1)}{3^4.5^4.7^4}+\frac{(3^4-1)(5^4-1)(7^4-1)}{3^4.5^4.7^4.11^4}$$

$$+\frac{(3^4-1)(5^4-1)(7^4-1)(11^4-1)}{3^4.5^4.7^4.11^4.13^4}+\ldots=1-\frac{96}{\pi^4},$$

where 3, 5, 7, 11, 13, 17, 19, 23, ... form the sequence of prime numbers

1430. (A. NARASINGA RAO):—The following experiment is intended to determine the instant of true noon at a given station. Discuss the accuracy attainable in practice taking into account errors both of reading and of adjustment:—

A fixed object moderately high up and casting a clear pointed shadow is selected and a drawing board arranged horizontally to catch the shadow. A plumb-line is suspended from the point of the object so as to graze the board at some point A. A few minutes before the expected instant mark the successive positions of the shadow on the board. These lie approximately on a straight line to which a perpendicular AX is now drawn through A. At noon, the shadow cast by the plumb-line coincides with AX.

1431. (V. RAMASWÁMI AIYAR):—If two triangles inscribed in a circle be such that the sides of either are pedal lines with respect to the vertices of the other, show that the six sides of the triangles touch a conic whose centre is the middle point of the join of the circum-centres of the triangles.

1432. (M. BHIMASENA RAO):—Given a triangle ABC, prove that the orthopolar ellipse of any point has double contact with the concentric conic inscribed in ABC.

1433. (V. RAMASWAMI AIYAR and M. BHIMASENA RAO:—Two quadrangles ABCD, A′B′C′D′ are such that the six sets of points like (BCA′D′) are orthocentric. Prove that the six meets of their corresponding sides (like BC, B′C′) and the six meets of their non-corresponding sides (like BC, A′D′) form a set of twelve points lying on a circle.

Show that the quadrangles are fixed when three vertices of the one and the non-corresponding vertex of the other (as A, B, C and D′) are given; and show also how, when one of these quadrangles is given, the other can be found.

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LIST OF JOURNALS RECEIVED

during March and April 1926.

1 Acta Mathematica, **46**, 3 and 4.*
2 American Journal of Math., **48**, 1 (2 copies).
3 American Math. Monthly, **33**, 1 (and Supplement)
4 Annales de lecole Normale Superieure, Dec. 1925.
5 Annals of Math. **27**, 2.
6 Astrophysical Journal, **62**, 5, **63**, 1.
7 Bulletin des Sciences Math., Jan. and Feb. 1925.
8 Bulletin of the Calcutta Math. Soc. **16**, 3 (2 copies).
9 Mathematical Gazette, **13**, 181, (3 copies).
10 Math. Annalen, **95**, 4.
11 Messenger of Mathematics, **55**, 7, 8 and 9.
12 Monthly Notices of the Royal Astronomical Society, **86**, 2.
13 Nature, **29**, 32 to 44.
14 Niew Archiv Voor Wiskunde (Nieuwe Opgaven Deel XV, 41—97)
15 Philosophical Magazine, **1**, (7th Series); 2 and 3.
16 Popular Astronomy, **34**, 2 and 3 (2 copies).
17 Proceedings of the Cambridge Philosophical Society, **22**, 6; **23**, 1.
18 Proceedings of the London Mathematical Society, **24**, 7 (3 copies.
19 Proceedings of the Royal Society, **110**, 754 and 755.
20 Rendiconti del circolo di Palermo, **49**, 2.
21 Revue Sunestrieke des publications Mathematiques, **31**, 1, 2.
22 Transactions of Cambridge Philosophical Society, **23**, 8.

Back numbers.

Nouvelles Annales de Math.

Series	1	Tome	**1 to 20.**
„	2	„	**1 to 20.**
„	3	„	**1 to 19.**
„	4	„	**1 to 20.**
„	5	„	**1 & 2.**

Books.

Bombay University Hand-book, 1925, 1926, 1927.

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* Numbers in black type refer to the Volume, and those in ordinary type to the specific number of the issue.

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